RECONSTRUCTION OF DEPTH AND NORMALS FROM INTERREFLECTIONS

Binh-Son Hua[1] *Tian-Tsong Ng*[2] *Kok-Lim Low*[1]

[1]National University of Singapore [2]Institute for Infocomm Research, Singapore

ABSTRACT

While geometry reconstruction has been extensively studied, several shortcomings still exist. First, traditional geometry reconstruction methods such as geometric or photometric stereo only recover either surface depth or normals. Second, such methods require calibration. Third, such methods cannot recover accurate geometry in the presence of interreflections. In order to address these problems in a single system, we propose an approach to reconstruct geometry from light transport data. Specifically, we investigate the problem of geometry reconstruction from interreflections in a light transport matrix. We show that by solving a system of polynomial equations derived directly from the interreflection matrix, both surface depth and normals can be fully reconstructed. Our system does not require projector-camera calibration, but only make use of a calibration object such as a checkerboard in the scene to pre-determine a few known points to simplify the polynomial solver. Our experimental results show that our system is able to reconstruct accurate geometry from interreflections up to a certain noise level. Our system is easy to set up in practice.

Index Terms— light transport, shape from interreflections, depth reconstruction, normal reconstruction

1. INTRODUCTION

Geometry reconstruction has been extensively studied in computer vision in the past decades. Reconstruction techniques such as geometric stereo and photometric stereo have greatly matured and have widely been used in both scientific and industrial applications. However, like many other computer vision techniques, previous reconstruction approaches only account for direct illumination and ignores an important lighting effect that often occurs in a scene: global illumination. Therefore, those techniques can only handle scenes in which interreflection or sub-surface scattering is absent. In order to improve robustness of geometry reconstruction, global illumination would need to be properly considered.

4D light transport is a general matrix representation that captures a scene observed in a set of varying illuminations. An entry in the matrix captures the out-going radiance at a scene point illuminated by a light source. It is also well-known that under Lambertian assumption, light transport matrix can be factorized into the first-bounce light transport matrix which captures direct illumination, and the interreflection matrix which captures illumination that bounces from a surface to another in the scene [1]. In computer graphics, several applications of light transport have been proposed such as relighting, dual photography, and radiometric compensation. However, in computer vision, light transport has not been received great attentions for tasks such as geometry reconstruction. Since light transport captures global illumination, it is of great interest to explore geometry reconstruction from such global illumination data.

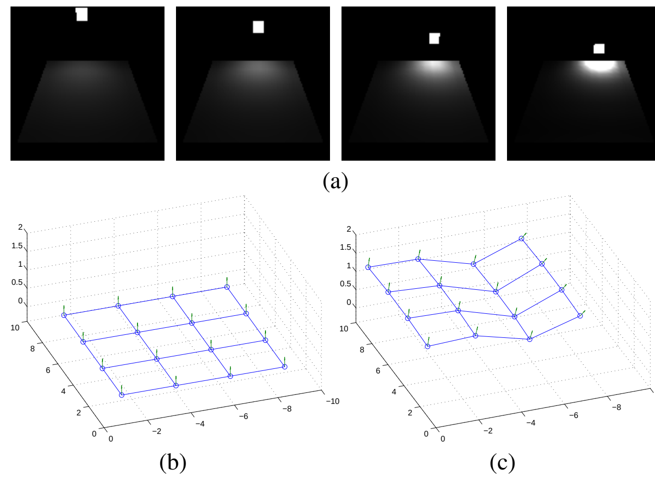


Fig. 1. (a) Synthetic light transport using radiosity. (b) Reconstructed points from exact data by form factor formula. (c) Reconstructed points from data by radiosity renderer.

In this work, we present a new approach to recover scene geometry from light transport. Our reconstruction is based on solving a system of polynomial equations derived directly from the interreflection matrix. We show that our method can reconstruct both surface depth and normals from interreflections. Our method does not require the projector and the camera to be calibrated. It also does not rely on orthographic assumption and planar constraints [2]. We only use a checkerboard pattern in the scene to pre-determine coordinates of a few points to bootstrap the solving of polynomial equations. Therefore, it can more easy to use in practice.

2. RELATED WORKS

In this section, we first discuss two classes of traditional reconstruction techniques, triangulation-based methods and photometric stereo methods. We then discuss about recent techniques that recover geometry in the presence of global illumination.

2.1. Conventional methods

Triangulation-based methods, e.g., geometric stereo and structured light scanning, has long been common approaches for geometry reconstruction. Geometric stereo is sometimes problematic since it relies on scene features such as corners to determine correspondences, which is not always robust. Structured light scanning projects special light patterns into the scene so that correspondences between the projector and the camera can be decoded in the post-process. However, while triangulation-based methods yields 3D coordinates of scene points, it does not compute surface normals directly. Surface normals can be found from derivatives of local surfaces which needs to be reconstructed for each neighborhood of scene points.

On the other hand, photometric stereo observes the scene under varying illumination with the camera view fixed. Based on surfaces illuminated by at least three different directional light sources, surface normals can be solved from a linear system. In contrast to triangulation-based methods, photometric stereo yields surface normals directly, but it does not compute 3D coordinates of surface points. 3D coordinates can be determined by integrating normal vectors. Since triangulation-based methods and photometric stereo reconstruct attributes of surfaces that are complementary to each other, it is of great interest to seek methods that can produce surface depth and normals at the same time. In this work, we propose such an approach that aims to reconstruct geometry from light transport.

Also, a common drawback of conventional geometric and photometric stereo is that calibration is necessary. Geometric stereo requires the camera to be calibrated while photometric stereo assumes directional light source and requires the directions of the light sources to be known. Some efforts has been done to relax the necessity of such calibration. For example, Basri and Jacobs [3] showed that surface normals can be recovered from uncalibrated photometric stereo up to a general bas-relief ambiguity. Recently, Yamazaki et al. [4] proposed the joint recovery of intrinsic and extrinsic parameters of both camera and projectors in a projector-camera setup. However, their method still requires the center of projection of both camera and projector to be known.

Extensions of photometric stereo to near point light source have also been proposed [5, 6]. In such setup, depth recovery can be incorporated into photometric stereo due to the modeling of light fall-off by the inverse squared law. However, while near point light source is more practical, these methods still require the location of the light sources to be known. Our system is more convenient as it does not require the calibration of the projector. The only object that we need is a checkerboard pattern put in the scene to help determine known points in the post-processing.

2.2. Hybrid methods

In this work, our proposed system jointly reconstructs surface depth and normals and hence can be regarded as a combination of geometric and photometric stereo in terms of output. In this aspect, several similar hybrid systems have been proposed in the past [7, 8, 9]. For example, Aliaga and Xu [7] proposed a self-calibration method that utilizes both geometric and photometric stereo. Holroyd et al. [8] combined multiple view reconstruction and phase shifting to recover complete 3D geometry and surface reflectance of a target object. Yoon et al. [9] suggested a non-linear optimization framework to recover geometry and reflectance from multiple view geometry, which requires a good initialization for the non-linear optimization. While our system is quite similar to these works, we explore geometry reconstruction from light transport data of a scene. This can be more convenient since light transport can also be at the same time utilized for other applications relighting and radiometric compensation. Our system also does not require explicit calibration as in [8].

2.3. Reconstruction in the presence of global illumination

While traditional reconstruction methods work well for Lambertian and mostly diffuse surfaces, they ignore an important effect that is commonly seen: global illumination. This strict assumption can limit accurate shape reconstruction when global illumination is strong, e.g., when light bounces within concave surfaces. It has been shown that photometric stereo tends to produce a shallower concave surface if interreflections are not taken into account [10].

In order to accurately reconstruct geometry in the presence of global illumination, two different strategies can be used. The first approach is to separate global illumination based on the principle proposed by Nayar et al. [11]. They show that since global illumination is a low-frequency effect, it is almost invariant to high-frequency illumination. Therefore, by using high-frequency light patterns, either binary or phase-shift patterns, it is possible to separate direct and global illumination. Since then, several methods have been proposed to make geometry reconstruction robust to global illumination [12, 13]. Gupta et al. [12] studied the relationship between projector defocus and global illumination and showed that such adverse effects can be separated and removed from the scene. Geometry can then be reconstructed from direct illumination. Gupta et al. [13] proposed a method to design structured light patterns that yield accurate correspondences in the presence of short-range and long-range global illumination. Gupta and Nayar [14] also suggested that phase shifting can be extended to include only high-frequency patterns so that reconstruction is robust to global illumination. Couture et al. [15] showed that random patterns can also be used to finding robust correspondences. However, methods based on explicitly removing global illumination and reconstructing geometry from residual direct illumination can still fail when signal-to-noise ratio of direct illumination is too low, e.g, as in translucent objects that have strong sub-surface scattering. Approaches that do not require explicit removal of global illumination do not have this drawback, but they need different pattern designs to handle different global illumination effects [13]. Furthermore, all these approaches are based on triangulation, which requires the light source and the camera to be fully calibrated.

Another approach to handle global illumination is to model it explicitly, which is also the approach we chose to follow. This class of methods can be useful when the scene is dominated by global illumination. Nayar et al. [10] proposed to refine surface normals obtained by photometric stereo using interreflection. Liu et al. [2] proposed to reconstruct geometry from the interreflection matrix. We note that the work in [2] is probably most related to ours. However, the authors assumed orthographic projection and did not properly handle the area term in the interreflection model. We show that our method is independent of the type of camera projection, and it can handle the area term properly by considering it as an unknown scalar in the system of polynomials.

In summary, we highlight three shortcomings from previous approaches. First, triangulation-based methods only recover surface depth while photometric stereo only recovers surface normals. Second, traditional geometric and photometric stereo require the acquisition system to be carefully calibrated. Hybrid methods are needed to jointly recover both surface depth and normals. Third, and more importantly, global illumination is often ignored, which can cause reconstruction surfaces to be shallower, as shown in [10]. As far as we know, there has been no single acquisition system that address such shortcomings altogether.

Therefore, in this work, we propose to build an acquisition system that is aimed to fill this gap. Our hybrid system can jointly recovers surface depth and normals. We explore how to reconstruct such depth and normals directly from interreflections in a light transport. Our system does not require orthographic assumption and planar constraints as in [2] and does not need calibration. We only use a checkerboard in the scene to determine a few known points in order to simplify the polynomial solver in the reconstruction. Therefore, our system is easier to implement and more convenient to use in practice.

3. INTERREFLECTIONS IN LIGHT TRANSPORT

The rendering equation that computes the out-going radiance L at scene point $\mathbf{x}$ to scene point $\mathbf{x}''$ can be written as

$$L(\mathbf{x}, \mathbf{x}'') = L_d(\mathbf{x}, \mathbf{x}'') + \int_{\mathbf{x}'} \mathbf{A}(\mathbf{x}', \mathbf{x}, \mathbf{x}'') L(\mathbf{x}', \mathbf{x}) d\mathbf{x}' \quad (1)$$

where $\mathbf{A}$ is the interreflection operator, L_d is the direct illumination from $\mathbf{x}$ to $\mathbf{x}''$. We define light transport operator $\mathbf{T}$ that captures the net effect of the whole light transport in the scene as follows.

$$L(\mathbf{x}, \mathbf{x}'') = \int_{\mathbf{x}'} \mathbf{T}(\mathbf{x}', \mathbf{x}, \mathbf{x}'') L_e(\mathbf{x}', \mathbf{x}) d\mathbf{x}', \quad (2)$$

where L_e is the emitted radiance from light sources. Similarly, we define the first-bounce light transport $\mathbf{F}$ which only stores direct illumination as

$$L_d(\mathbf{x}, \mathbf{x}'') = \int_{\mathbf{x}'} \mathbf{F}(\mathbf{x}', \mathbf{x}, \mathbf{x}'') L_e(\mathbf{x}', \mathbf{x}) d\mathbf{x}'. \quad (3)$$

As we assume Lambertian surfaces, the rendering equation becomes the radiosity equation. Since the out-going radiance is the same for all directions determined by $\mathbf{x}''$, we drop the outgoing direction $\mathbf{x}''$ and simply store radiosity $\pi L(\mathbf{x}, \mathbf{x}')$ at each surface point $\mathbf{x}$. Numerically, a light transport matrix $\mathbf{T}$ can be represented by

$$\mathbf{T} = (\mathbf{I} - \mathbf{A})^{-1} \mathbf{F} \quad (4)$$

where $\mathbf{I}$ is the identity matrix. Since all surfaces are Lambertian, first-bounce $\mathbf{F}$ and inverse light transport $\mathbf{T}^{-1}$ can be computed from light transport $\mathbf{T}$ as in [1]. The interreflection matrix $\mathbf{A}$ can be obtained by

$$\mathbf{A} = \mathbf{I} - \mathbf{F}\mathbf{T}^{-1}. \quad (5)$$

Since the interreflection matrix $\mathbf{A}$ captures how much illumination bounces from a surface to another in the scene, it is possible to utilize such information for geometry reconstruction. We show how it can be done in the following section.

4. OUR METHOD

4.1. Polynomial equations from interreflections

Each element $\mathbf{A}_{i \to j}$ (represented as matrix entry $\mathbf{A}_{ji}$) captures how radiosity from a source surface patch i contributes to a target patch j and can be written as:

$$\mathbf{A}_{i \to j} = \mathbf{k}_j \mathbf{G}_{i \leftrightarrow j} \Delta_i \quad (6)$$

where $\mathbf{k}_j$ is the albedo of patch j, Δ_i the area of patch i, and $\mathbf{G}_{i \leftrightarrow j} = \mathbf{G}_{j \leftrightarrow i}$ the geometric factor between patch i and patch j:

$$\mathbf{G}_{i \leftrightarrow j} = \frac{\mathbf{n}_i^\top (\mathbf{x}_i - \mathbf{x}_j) \mathbf{n}_j^\top (\mathbf{x}_i - \mathbf{x}_j)}{\|\mathbf{x}_i - \mathbf{x}_j\|^4} \quad (7)$$

where $\mathbf{x}$ and $\mathbf{n}$ denote the location and orientation of a patch, respectively. If patch i is visible in the camera view, its area can be approximated as:

$$\Delta_i = \Delta_{\text{pixel}} \frac{\|\mathbf{c} - \mathbf{x}_i\|}{\mathbf{n}_i^\top (\mathbf{c} - \mathbf{x}_i)} \quad (8)$$

where $\mathbf{c}$ is the camera location and Δ_{pixel} is the area of the pixel that contains patch i.

It is easy to see that the interreflection matrix $\mathbf{A}$ captures albedo, location, and orientation of geometric points in the scene. Our goal is to reconstruct the location and orientation of the geometry from $\mathbf{A}$. However, solving the complete geometry from $\mathbf{A}$ can be very challenging because interreflection equations are non-linear and there are a large number of unknowns. To make the problem tractable, we assume a set of known points Q in the scene and try to reconstruct the set of unknown points P from the interreflections between P and Q.

Consider a pair of points $\mathbf{p}_i$ and $\mathbf{q}_j$ where $i \in P$ and $j \in Q$. We would like to reconstruct the albedo, location, and orientation of $\mathbf{p}_i$ from its interreflection with $\mathbf{q}_j$, which are captured by entries $\mathbf{A}_{i \to j}$ and $\mathbf{A}_{j \to i}$ in the interreflection matrix.

Consider $\mathbf{A}_{i \to j}$. We observe that equation $\mathbf{A}_{i \to j}$ is almost a polynomial except the area term Δ_i that depends on the foreshortening of the patch to the camera view. We now show how to formulate $\mathbf{A}_{i \to j}$ into a polynomial.

For simplicity, we first drop index i since we are going to fix i and only consider $\mathbf{A}_{i \to j}$ for varying j. Therefore, we rewrite Equation 6 as

$$\mathbf{A}_j = \mathbf{k}_j \mathbf{G}_{i \leftrightarrow j} \Delta \quad (9)$$

Let $\mathbf{a}_j = \mathbf{k}_j \Delta$. We further assume that $\mathbf{k}_j$ is invariant for points $j \in Q$ where $\mathbf{A}_j > 0$. This is a reasonable assumption since we can group points that have similar albedos together into group Q. This allows us to model $\mathbf{a}_j$ as a single scalar variable $a = \mathbf{a}_j$ for all $j \in Q$. Multiplying a with the orientation $\mathbf{n}$ to obtain $\mathbf{m} = a\mathbf{n}$, the radiosity from $\mathbf{q}_j$ to $\mathbf{p}_i$ can be written as:

$$\mathbf{A}_j = \frac{\mathbf{m}^\top (\mathbf{x} - \mathbf{x}_j) \mathbf{n}_j^\top (\mathbf{x} - \mathbf{x}_j)}{\|\mathbf{x} - \mathbf{x}_j\|^4} \quad (10)$$

which is a polynomial equation in which the unknowns are a 6-DOF vector $(\mathbf{x}, \mathbf{m})$. We now propose an algorithm to solve $(\mathbf{x}, \mathbf{m})$.

4.2. Algorithm to recover location and orientation

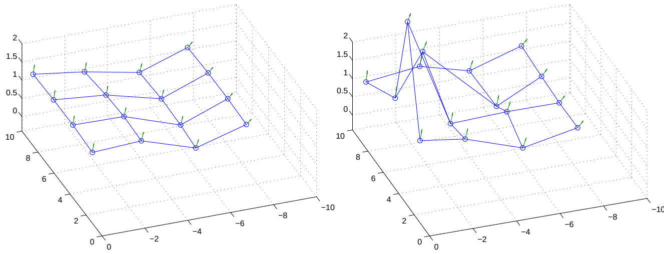
Equation 10 suggests that at least six points in Q are necessary to recover each point $\mathbf{p}_i$ separately. The equations can be easily built given the entries $\mathbf{A}_{i \to j}$ for $j \in Q$. Notice that we do not make use of $\mathbf{A}_{i \to j}$ by fixing j and varying i in group P because it is often less practical to assume that the area term Δ_i is constant for different i.

We build an algebraic polynomial solver based on Groebner basis to solve $(\mathbf{x}, \mathbf{m})$. We observe that the solutions given by the algebraic solver are very close to the ground truth, and can be further refined by a non-linear iterative solver when necessary. In general, the algorithm to reconstruct $(\mathbf{x}, \mathbf{m})$ at each point $\mathbf{p}_i$ is as follows.

1. Randomly select six points $\mathbf{q}_j$ s.t. $j \in Q$ and $\mathbf{A}_{i \to j} > 0$.
2. Reconstruct $(\mathbf{x}, \mathbf{m})$ using an algebraic polynomial solver.
3. Compute the residuals from the polynomial equations and repeat the above steps N times. Take $(\mathbf{x}, \mathbf{m})$ that has the lowest residual.
4. Refine $(\mathbf{x}, \mathbf{m})$ with all points $\mathbf{q}_j$ in Q by a non-linear iterative solver.

4.3. Implementation

In practice, we implement the above framework with the following assumptions. In Step 1, we assume points in set Q to be planar. Locations and orientations of points on a plane can be easily determined by a simple camera calibration. In Step 2, we translate known points



(a) Variance 10^{-2} (b) Variance 10^{-1}

Fig. 2. Reconstruction results with noise variance 10^{-2} and 10^{-1} added to input images.

to plane $z = 0$ and orient the plane towards positive Z-axis. We note that this simplifies the Groebner basis of the system of polynomials to a set of 36 monomials. Positioning the plane at other locations can make the system of polynomials more challenging to solve. For example, letting the plane be at $z = \alpha$ that $\alpha \neq 0$ results in a Groebner basis that has 106 monomials. We implement a floating-point polynomial solver based on the action matrix approach. Since there may have several solutions, those that violates visibility constraints are discarded in advance before proceeding to compute residuals. Step 3 is very similar to RANSAC [16]. However, here only a few iterations of the first two steps are needed since the result can be refined in Step 4. We use Levenberg-Marquardt optimization [17] in Step 4.

5. EXPERIMENTS

We test our algorithm with a synthetic scene rendered by direct form factor calculation and a progressive radiosity algorithm. We use 16 area light sources to individually illuminate a known plane Q. The light sources are distributed uniformly on an unknown plane P and our goal is to reconstruct the locations and orientations of the light sources. For simplicity we only render direct illumination and set albedos of scene objects to one. Therefore, the radiance observed at plane Q can be directly used to find the locations and orientations of light sources on P.

Figure 1 demonstrates that our algorithm can successfully reconstruct the locations and orientations of each light sources. We note that our synthetic example is sufficient to test our reconstruction from the system of polynomials. While our algorithm can work with both data from exact form factor and data generated by a radiosity renderer in this example, we did notice a slight shift in the geometry reconstructed from the later as compared to the groundtruth. This can be due to inaccuracy of the intensity values generated by radiosity methods.

In practice, the captured images can be subject to noise. In order to test how our method behaves to noise in this synthetic scenario, we proceed to add Gaussian noise to observed pixel values. Figure 2 shows that our solver can tolerate a certain amount of noise with variance up to 10^{-1}.

We acknowledge that since our method relies on radiometric values, i.e., radiance, and numerical solvers for reconstruction, our recovered geometry can be susceptible to noise and may not be as accurate as traditional methods that bases on triangulation.

6. CONCLUSIONS

We proposed a novel approach to acquire geometry from interreflections. A system of polynomial equations is established directly from the interreflection matrix and we show that by solving this system of polynomial equations, the geometry of the scene, i.e., surface depth and normal vectors, can be jointly reconstructed. Our experimental results demonstrated that our method works well with synthetic datasets up to a certain noise level. Our system is convenient since it does not require calibration.

Our system is limited by the following factors. First, while projector and camera calibration are not needed, a planar checkerboard must be placed in the scene and interact with scene objects in order to simplify the polynomial system. This can cause the arrangement of objects in the scene to be not flexible. Second, our system can be susceptible to noise. The floating-point implementation of the solver of polynomial equations may return wrong solutions when the input data is perturbed by a small amount of noise. Third, our model is based on Lambertian assumption. In practice, this assumption may not be always true. Surfaces in the scene can be up to some certain degrees of glossiness, which violates the interreflection model and causes the system to fail to reconstruct the geometry. Finally, since we rely on acquiring light transport and solving polynomials for geometry reconstruction, our system is not fast enough for real-time reconstruction.

From this study, we recognize several open problems for future research. A potential direction is to design reconstruction methods for more general materials, e.g., glossy or sub-surface scattering surfaces. It is more challenging to fully model such effects than to model diffuse interreflections. Moreover, extracting the global illumination matrix in such cases can be more difficult if the first-bounce matrix is not given. One of the first works in this direction, e.g., shape from translucent surfaces, has been proposed in [18]. Another potential direction can be to investigate the stability of the polynomial solver used in our approach. In this work, we only used the simplest form of the floating-point implementation of a polynomial solver. We hypothesize that the solver can perform better if stability approaches can be added [19]. Finally, it is of great interest to study fast light transport acquisition to accelerate the data capturing stage and make the system more practical. We also would like to perform more physical experiments to further test our whole proposed pipeline thoroughly, since in this work we only present synthetic examples.

7. REFERENCES

[1] Steven M. Seitz, Yasuyuki Matsushita, and Kiriakos N. Kutulakos, "A theory of inverse light transport," in *International Conference on Computer Vision (ICCV)*, 2005.

[2] Siying Liu, Tian-Tsong Ng, and Yasuyuki Matsushita, "Shape from second-bounce of light transport," in *European Conference on Computer Vision (ECCV)*, September 2010.

[3] Ronen Basri, David Jacobs, and Ira Kemelmacher, "Photometric stereo with general, unknown lighting," *International Journal of Computer Vision (IJCV)*, vol. 72, pp. 239–257, 2007.

[4] Shuntaro Yamazaki, Masaaki Mochimaru, and Takeo Kanade, "Simultaneous self-calibration of a projector and a camera using structured light," in *Computer Vision and Pattern Recognition Workshops (CVPRW)*, June 2011, pp. 60 –67.

[5] Yuji Iwahori, Hidezumi Sugie, and Naohiro Ishii, "Reconstructing shape from shading images under point light source illumination," in *International Conference on Pattern Recognition (ICPR)*, June 1990, vol. 1, pp. 83 –87.

[6] Byungil Kim and Peter Burger, "Depth and shape from shading using the photometric stereo method," *CVGIP: Image Understanding*, vol. 54, no. 3, pp. 416 – 427, 1991.

[7] Daniel G. Aliaga and Yi Xu, "Photogeometric structured light: A self-calibrating and multi-viewpoint framework for accurate 3d modeling," in *Computer Vision and Pattern Recognition (CVPR)*, 2008.

[8] Michael Holroyd, Jason Lawrence, and Todd Zickler, "A coaxial optical scanner for synchronous acquisition of 3d geometry and surface reflectance," *ACM Transactions on Graphics (Proc. ACM SIGGRAPH)*, 2010.

[9] Kuk-Jin Yoon, Emmanuel Prados, and Peter Sturm, "Joint estimation of shape and reflectance using multiple images with known illumination conditions," *International Journal of Computer Vision (IJCV)*, vol. 86, no. 2-3, pp. 192–210, January 2010.

[10] Shree K. Nayar, Katsushi Ikeuchi, and Takeo Kanade, "Shape from interreflections," *International Journal of Computer Vision (IJCV)*, vol. 6, pp. 173–195, August 1991.

[11] Shree K. Nayar, Gurunandan Krishnan, Michael D. Grossberg, and Ramesh Raskar, "Fast Separation of Direct and Global Components of a Scene using High Frequency Illumination," *ACM Transactions on Graphics (Proc. ACM SIGGRAPH)*, July 2006.

[12] Mohit Gupta, Yuandong Tian, Srinivasa Narasimhan, and Li Zhang, "A combined theory of defocused illumination and global light transport," *International Journal of Computer Vision (IJCV)*, vol. 98, pp. 146–167, 2012.

[13] Mohit Gupta, Amit Agrawal, Ashok Veeraraghavan, and Srinivasa Narasimhan, "A practical approach to 3d scanning in the presence of interreflections, subsurface scattering and defocus," *International Journal of Computer Vision (IJCV)*, pp. 1–23, 2012.

[14] Mohit Gupta and Shree K. Nayar, "Micro phase shifting," in *Computer Vision and Pattern Recognition (CVPR)*, June 2012, pp. 813 –820.

[15] Vincent Couture, Nicolas Martin, and Sebastien Roy, "Unstructured light scanning to overcome interreflections," in *International Conference on Computer Vision (ICCV)*, November 2011, pp. 1895 –1902.

[16] Martin A. Fischler and Robert C. Bolles, "Random sample consensus: a paradigm for model fitting with applications to image analysis and automated cartography," *Commun. ACM*, vol. 24, no. 6, pp. 381–395, June 1981.

[17] Jorge J. Moré, "The levenberg-marquardt algorithm: Implementation and theory," in *Numerical Analysis*, G.A. Watson, Ed., vol. 630 of *Lecture Notes in Mathematics*, pp. 105–116. Springer Berlin Heidelberg, 1978.

[18] Chika Inoshita, Yasuhiro Mukaigawa, Yasuyuki Matsushita, and Yasushi Yagi, "Shape from single scattering for translucent objects," in *European Conference on Computer Vision (ECCV)*, September 2012.

[19] Martin Byrod, Klas Josephson, and Kalle Astrom, "Fast and stable polynomial equation solving and its application to computer vision," *International Journal of Computer Vision (IJCV)*, vol. 84, pp. 237–256, 2009.